

IDENTITIES INVOLVING (DOUBLY) SYMMETRIC POLYNOMIALS AND INTEGRALS OVER GRASSMANNIANS

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ABSTRACT. We obtain identities for symmetric and doubly symmetric polynomials. These identities provide a way of handling expressions appearing in the Atiyah-Bott-Berline-Vergne formula for Grassmannians. As a corollary, we obtain formulas for integrals over Grassmannians of characteristic classes of the tautological bundles.

1. INTRODUCTION

Let $\lambda_1, \dots, \lambda_n$ be n distinct values. The Lagrange interpolation formula says that a polynomial $P(x)$ of degree not greater than $n - 1$ in one variable can be written in the following:

$$P(x) = \sum_{i=1}^n P(\lambda_i) L_i(x),$$

where

$$L_i(x) = \prod_{j \neq i} \frac{x - \lambda_j}{\lambda_i - \lambda_j}.$$

This formula not only provides a means of polynomial approximation, but also plays a significant role in establishing combinatorial identities in many variables including identities on symmetric polynomials. For instance, given n distinct values $\lambda_1, \lambda_2, \dots, \lambda_n$ ($n \geq 2$) and an integer $m \geq 0$, we have the following identity:

$$(1) \quad \sum_{i=1}^n \frac{\lambda_i^m}{\prod_{j \neq i} (\lambda_i - \lambda_j)} = h_{m-n+1}(\lambda_1, \dots, \lambda_n),$$

where h_k is the k -th homogeneous symmetric polynomial in n variables, which is defined to be zero for $k < 0$. The identity can be easily derived from the Lagrange interpolation formula (see [8, Theorem 2.2] for more details). In particular, for $m = 0, 1, \dots, n - 1$, the left hand side of (1) is independent to $\lambda_1, \dots, \lambda_n$. It is equal to zero if $m < n - 1$ and one if $m = n - 1$. This implies the following result.

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Proposition 1. *Let $P(x)$ be a polynomial of degree not greater than $n - 1$ in one variable. Then the sum*

$$\sum_{i=1}^n \frac{P(\lambda_i)}{\prod_{j \neq i} (\lambda_i - \lambda_j)} = c_n,$$

where c_n is the coefficient of x^{n-1} in the polynomial $P(x)$.

The proof of Proposition 1 is elementary. It is a direct corollary of the Lagrange interpolation formula.

The first goal of this paper is to give a generalization of Proposition 1 for multivariate symmetric polynomials. Throughout we always assume that all polynomials are over the field of rational numbers or any field of characteristic zero. For convenience, we shall write $[n]$ for the set $\{1, 2, \dots, n\}$. Let $\lambda_1, \dots, \lambda_n$ be n distinct values. For each subset $I = \{i_1, \dots, i_k\} \subset [n]$, we denote by $\lambda_I = (\lambda_{i_1}, \dots, \lambda_{i_k})$ and $I^c = [n] \setminus I$. Recall that a polynomial $P(x_1, \dots, x_k)$ is said to be symmetric if it is invariant under permutations of x_1, \dots, x_k . We obtain the following result.

Theorem 1. *Let $P(x_1, \dots, x_k)$ be a symmetric polynomial of degree not greater than $k(n - k)$ in k variables ($k < n$). Then the sum*

$$\sum_{I \subset [n], |I|=k} \frac{P(\lambda_I)}{\prod_{i \in I} \prod_{j \in I^c} (\lambda_i - \lambda_j)} = \frac{c(k, n)}{k!},$$

where $c(k, n)$ is the coefficient of $x_1^{n-1} \dots x_k^{n-1}$ in the polynomial

$$P(x_1, \dots, x_k) \prod_{i=1}^k \prod_{j \neq i} (x_i - x_j).$$

More generally, we also obtain an identity involving doubly symmetric polynomials. Recall that a polynomial $P(x_1, \dots, x_k, y_1, \dots, y_{n-k})$ is said to be doubly symmetric if it is invariant under permutations of x_1, \dots, x_k and permutations of y_1, \dots, y_{n-k} respectively. We obtain the following result.

Theorem 2. *Let $P(x_1, \dots, x_k, y_1, \dots, y_{n-k})$ be a doubly symmetric polynomial of degree not greater than $k(n - k)$. Then the sum*

$$\sum_{I \subset [n], |I|=k} \frac{P(\lambda_I, \lambda_{I^c})}{\prod_{i \in I} \prod_{j \in I^c} (\lambda_i - \lambda_j)} = \frac{d(k, n)}{k!(n - k)!},$$

where $d(k, n)$ is the coefficient of $x_1^{n-1} \dots x_k^{n-1} y_1^{n-1} \dots y_{n-k}^{n-1}$ in the polynomial

$$P(x_1, \dots, x_k, y_1, \dots, y_{n-k}) \prod_{i=1}^k \prod_{j \neq i} (x_i - x_j) \prod_{i=1}^{n-k} \prod_{j \neq i} (y_i - y_j) \prod_{i=1}^{n-k} \prod_{j=1}^k (y_i - x_j).$$

The second goal of this paper is to give a way of dealing with integrals over Grassmannians. The idea is as follows. Localization in equivariant cohomology allows us to express integrals in terms of some data attached to the fixed points of a torus action. In particular, for Grassmannians, we obtain interesting formulas with nontrivial relations involving rational functions. Let $G(k, n)$ be the Grassmannian of k -dimensional linear spaces in \mathbb{C}^n . Consider the following integrals:

$$\int_{G(k, n)} \Phi(\mathcal{S}) \quad , \quad \int_{G(k, n)} \Psi(\mathcal{Q}) \quad , \quad \int_{G(k, n)} \Delta(\mathcal{S}, \mathcal{Q}),$$

where $\Phi(\mathcal{S}), \Psi(\mathcal{Q})$ are respectively characteristic classes of the tautological sub-bundle \mathcal{S} and quotient bundle \mathcal{Q} on the Grassmannian $G(k, n)$, and $\Delta(\mathcal{S}, \mathcal{Q})$ is a characteristic class of both \mathcal{S} and \mathcal{Q} .

Using localization in equivariant cohomology, Weber [13] and Zielenkiewicz [15] presented a way of expressing the integrals as iterated residues at infinity of holomorphic functions. However, there is no guarantee expressions appearing in the Atiyah-Bott-Berline-Vergne formula for Grassmannians are in fact constants rather than rational functions. Our identities provide a way of handling such expressions.

Theorem 3. *Suppose that $\Phi(\mathcal{S})$ is represented by a symmetric polynomial $P(x_1, \dots, x_k)$ of degree not greater than $k(n - k)$ in k variables x_1, \dots, x_k which are the Chern roots of \mathcal{S} and $\Psi(\mathcal{Q})$ is represented by a symmetric polynomial $Q(y_1, \dots, y_{n-k})$ of degree not greater than $k(n - k)$ in $n - k$ variables y_1, \dots, y_{n-k} which are the Chern roots of \mathcal{Q} . We then have the following statements:*

(1) *The integral*

$$\int_{G(k, n)} \Phi(\mathcal{S}) = (-1)^{k(n-k)} \frac{c(k, n)}{k!},$$

where $c(k, n)$ is the coefficient of $x_1^{n-1} \dots x_k^{n-1}$ in the polynomial

$$P(x_1, \dots, x_k) \prod_{i=1}^k \prod_{j \neq i} (x_i - x_j).$$

(2) *The integral*

$$\int_{G(k, n)} \Psi(\mathcal{Q}) = \frac{c(k, n)}{(n - k)!},$$

where $c(k, n)$ is the coefficient of $y_1^{n-1} \dots y_{n-k}^{n-1}$ in the polynomial

$$Q(y_1, \dots, y_{n-k}) \prod_{i=1}^{n-k} \prod_{j \neq i} (y_i - y_j).$$

Theorem 4. *Suppose that $\Delta(\mathcal{S}, \mathcal{Q})$ is represented by a doubly symmetric polynomial*

$$P(x_1, \dots, x_k, y_1, \dots, y_{n-k})$$

of degree not greater than $k(n-k)$ in n variables which are the Chern roots of \mathcal{S} and \mathcal{Q} respectively. We then have the integral

$$\int_{G(k,n)} \Delta(\mathcal{S}, \mathcal{Q}) = (-1)^{k(n-k)} \frac{d(k,n)}{k!(n-k)!},$$

where $d(k,n)$ is the coefficient of $x_1^{n-1} \dots x_k^{n-1} y_1^{n-1} \dots y_{n-k}^{n-1}$ in the polynomial

$$P(x_1, \dots, x_k, y_1, \dots, y_{n-k}) \prod_{i=1}^k \prod_{j \neq i} (x_i - x_j) \prod_{i=1}^{n-k} \prod_{j \neq i} (y_i - y_j) \prod_{i=1}^{n-k} \prod_{j=1}^k (y_i - x_j).$$

As a special case, if $\Delta(\mathcal{S}, \mathcal{Q}) = \Phi(\mathcal{S})\Psi(\mathcal{Q})$ is represented by the product of two symmetric polynomials $P(x_1, \dots, x_k)$ and $Q(y_1, \dots, y_{n-k})$, then we have the following corollary.

Corollary 1. *Suppose that $\deg(P) + \deg(Q) \leq k(n-k)$. The integral*

$$\int_{G(k,n)} \Phi(\mathcal{S})\Psi(\mathcal{Q}) = (-1)^{k(n-k)} \frac{d(k,n)}{k!(n-k)!},$$

where $d(k,n)$ is the coefficient of $x_1^{n-1} \dots x_k^{n-1} y_1^{n-1} \dots y_{n-k}^{n-1}$ in the polynomial

$$P(x_1, \dots, x_k)Q(y_1, \dots, y_{n-k}) \prod_{i=1}^k \prod_{j \neq i} (x_i - x_j) \prod_{i=1}^{n-k} \prod_{j \neq i} (y_i - y_j) \prod_{i=1}^{n-k} \prod_{j=1}^k (y_i - x_j).$$

The rest of the paper is organized as follows: The identities are proved in Section 2. Section 3 is to give a review of equivariant cohomology and the proof of Theorem 3.

2. PROOF OF THE IDENTITIES

We shall use X to denote the set $\{x_1, \dots, x_k\}$ or the n -tuple (x_1, \dots, x_k) , and Y to denote the set $\{y_1, \dots, y_{n-k}\}$ or the $(n-k)$ -tuple (y_1, \dots, y_{n-k}) . We denote by

$$Y - X = \prod_{i=1}^{n-k} \prod_{j=1}^k (y_i - x_j).$$

For $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_{n-k}\}$, we denote by $\lambda_I = (\lambda_{i_1}, \dots, \lambda_{i_k})$, $\lambda_J = (\lambda_{j_1}, \dots, \lambda_{j_{n-k}})$, and

$$\lambda_I - \lambda_J = \prod_{i \in I} \prod_{j \in J} (\lambda_i - \lambda_j).$$

Proof of Theorem 1. Set

$$F(X) = P(X) \prod_{i=1}^k \prod_{j \neq i} (x_i - x_j).$$

By the assumption, the degree of $F(X)$ is not greater than $k(n-k) + k(k-1) = k(n-1)$. Since P is symmetric, so F is also. By the division algorithm for multivariate

polynomials (see [7, Theorem 3]), there exist the polynomials $F_i(X), i = 1, \dots, k$ and $R(X)$ such that

$$R(X) = F(X) - \sum_{i=1}^k F_i(X) \prod_{j=1}^n (x_i - \lambda_j),$$

and all partial degrees of $R(X)$ are not greater than $n-1$. By the Lagrange interpolation formula, we have

$$R(X) = \sum_{i_1=1}^n R(\lambda_{i_1}, x_2, \dots, x_k) L_{i_1}(x_1).$$

By the Lagrange interpolation formula for the polynomials $R(\lambda_{i_1}, x_2, \dots, x_k)$, we have

$$R(X) = \sum_{i_1=1}^n \sum_{i_2=1}^n R(\lambda_{i_1}, \lambda_{i_2}, x_3, \dots, x_k) L_{i_1}(x_1) L_{i_2}(x_2).$$

So on, we have

$$R(X) = \sum_{i_1, \dots, i_k=1}^n R(\lambda_I) \prod_{l=1}^k L_{i_l}(x_l).$$

For each $I = \{i_1, \dots, i_k\}$, we have $R(\lambda_I) = F(\lambda_I)$, and if $i_s = i_t$ for some $s \neq t$, then $R(\lambda_I) = 0$. Since the degree of $F(X)$ is not greater than $k(n-1)$, so the coefficient of $x_1^{n-1} \dots x_k^{n-1}$ in $R(X)$ is equal to that in $F(X)$. Thus the coefficient of $x_1^{n-1} \dots x_k^{n-1}$ in $F(X)$ is equal to

$$k! \sum_{I \subset [n], |I|=k} \frac{F(\lambda_I)}{\prod_{i \in I} \prod_{j \neq i} (\lambda_i - \lambda_j)}.$$

For each $I = \{i_1, \dots, i_k\} \subset [n]$, we have

$$F(\lambda_I) = P(\lambda_I) \prod_{i \in I} \prod_{j \in I, j \neq i} (\lambda_i - \lambda_j),$$

and

$$\prod_{i \in I} \prod_{j \neq i} (\lambda_i - \lambda_j) = \prod_{i \in I} \prod_{j \in I^c} (\lambda_i - \lambda_j) \prod_{i \in I} \prod_{j \in I, j \neq i} (\lambda_i - \lambda_j).$$

This implies that the coefficient of $x_1^{n-1} \dots x_k^{n-1}$ in $F(X)$ is equal to

$$k! \sum_{I \subset [n], |I|=k} \frac{P(\lambda_I)}{\prod_{i \in I} \prod_{j \in I^c} (\lambda_i - \lambda_j)}.$$

Theorem 1 is proved as desired. \square

Remark 1. If $P(x_1, \dots, x_k)$ is a symmetric polynomial whose partial degrees are not greater than $n - k$, then we have the following formula, which is proved by Chen and Louck [8, Theorem 2.1],

$$P(X) = \sum_{I \subset [n], |I|=k} P(\lambda_I) \frac{\prod_{x_i \in X} \prod_{j \in I^c} (x_i - \lambda_j)}{\prod_{i \in I} \prod_{j \in I^c} (\lambda_i - \lambda_j)}.$$

Using this interpolation formula, we get

$$\sum_{I \subset [n], |I|=k} \frac{P(\lambda_I)}{\prod_{i \in I} \prod_{j \in I^c} (\lambda_i - \lambda_j)} = d(k, n),$$

where $d(k, n)$ is the coefficient of $x_1^{n-k} \dots x_k^{n-k}$ in $P(x_1, \dots, x_k)$. Indeed, this is a special case of Theorem 1. It is known that $k!$ is the coefficient of $x_1^{k-1} \dots x_k^{k-1}$ in the polynomial

$$\prod_{i=1}^k \prod_{j \neq i} (x_i - x_j).$$

This is proved in [14]. If the partial degrees of $P(x_1, \dots, x_k)$ are not greater than $n - k$, then we get

$$c(n, k) = d(k, n)k!.$$

Proof of Theorem 2. Set

$$F(X, Y) = P(X, Y) \prod_{i=1}^k \prod_{j \neq i} (x_i - x_j) \prod_{i=1}^{n-k} \prod_{j \neq i} (y_i - y_j) (Y - X).$$

By the assumption, the degree of $F(X, Y)$ is not greater than

$$k(n - k) + k(k - 1) + (n - k)(n - k - 1) + k(n - k) = n(n - 1).$$

By the division algorithm for multivariate polynomials (see [7, Theorem 3]), there exist the polynomials $F_i(X, Y), i = 1, \dots, k, G_j(X, Y), j = 1, \dots, n - k$, and $R(X, Y)$ such that

$$R(X, Y) = F(X, Y) - \sum_{i=1}^k F_i(X, Y) \prod_{l=1}^n (x_i - \lambda_l) - \sum_{j=1}^{n-k} G_j(X, Y) \prod_{l=1}^n (y_j - \lambda_l),$$

and all partial degrees of $R(X, Y)$ are not greater than $n - 1$. By the Lagrange interpolation formula as in the proof of Theorem 1, we have

$$R(X, Y) = \sum_{i_1, \dots, i_k, j_1, \dots, j_{n-k}=1}^n R(\lambda_{I}, \lambda_J) \prod_{l=1}^k L_{i_l}(x_l) \prod_{l=1}^{n-k} L_{j_l}(y_l).$$

For each $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_{n-k}\}$, we have

$$R(\lambda_I, \lambda_J) = F(\lambda_I, \lambda_J),$$

and if $i_s = i_t$ or $j_s = j_t$ or $i_s = j_t$ for some $s \neq t$, then $R(\lambda_I, \lambda_J) = 0$. Since the degree of $F(X, Y)$ is not greater than $n(n-1)$, so the coefficient of $x_1^{n-1} \dots x_k^{n-1} y_1^{n-1} \dots y_{n-k}^{n-1}$ in $R(X, Y)$ is equal to that in $F(X, Y)$. Thus the coefficient of $x_1^{n-1} \dots x_k^{n-1} y_1^{n-1} \dots y_{n-k}^{n-1}$ in $F(X, Y)$ is equal to

$$k!(n-k)! \sum_{I \subset [n], |I|=k} \frac{F(\lambda_I, \lambda_{I^c})}{\prod_{i \in I} \prod_{j \neq i} (\lambda_i - \lambda_j) \prod_{i \in I^c} \prod_{j \neq i} (\lambda_i - \lambda_j)}.$$

For each $I = \{i_1, \dots, i_k\} \subset [n]$, we have

$$\begin{aligned} F(\lambda_I, \lambda_{I^c}) &= P(\lambda_I, \lambda_{I^c}) \prod_{i \in I} \prod_{j \in I, j \neq i} (\lambda_i - \lambda_j) \prod_{i \in I^c} \prod_{j \in I^c, j \neq i} (\lambda_i - \lambda_j) (\lambda_{I^c} - \lambda_I) \\ &= P(\lambda_I, \lambda_{I^c}) \prod_{i \in I} \prod_{j \in I, j \neq i} (\lambda_i - \lambda_j) \prod_{i \in I^c} \prod_{j \in I^c, j \neq i} (\lambda_i - \lambda_j), \end{aligned}$$

and

$$\prod_{i \in I} \prod_{j \neq i} (\lambda_i - \lambda_j) = \prod_{i \in I} \prod_{j \in I^c} (\lambda_i - \lambda_j) \prod_{i \in I} \prod_{j \in I, j \neq i} (\lambda_i - \lambda_j).$$

This implies that the coefficient of $x_1^{n-1} \dots x_k^{n-1} y_1^{n-1} \dots y_{n-k}^{n-1}$ in $F(X, Y)$ is equal to

$$k! \sum_{I \subset [n], |I|=k} \frac{P(\lambda_I, \lambda_{I^c})}{\prod_{i \in I} \prod_{j \in I^c} (\lambda_i - \lambda_j)}.$$

Theorem 2 is proved as desired. \square

Remark 2. Theorem 2 is more general than Theorem 1. Indeed, if $P(x_1, \dots, x_k)$ is a symmetric polynomial, then it is also a doubly symmetric polynomial. Theorem 2 says that the sum

$$\sum_{I \subset [n], |I|=k} \frac{P(\lambda_I)}{\prod_{i \in I} \prod_{j \in I^c} (\lambda_i - \lambda_j)} = \frac{d(k, n)}{k!(n-k)!},$$

where $d(k, n)$ is the coefficient of $x_1^{n-1} \dots x_k^{n-1} y_1^{n-1} \dots y_{n-k}^{n-1}$ in the polynomial

$$P(x_1, \dots, x_k) \prod_{i=1}^k \prod_{j \neq i} (x_i - x_j) \prod_{i=1}^{n-k} \prod_{j \neq i} (y_i - y_j) \prod_{i=1}^{n-k} \prod_{j=1}^k (y_i - x_j).$$

It is known that $(n-k)!$ is the coefficient of $y_1^{n-k-1} \dots y_{n-k}^{n-k-1}$ in the polynomial

$$\prod_{i=1}^{n-k} \prod_{j \neq i} (y_i - y_j).$$

This is proved in [14]. Thus $(n - k)!$ is also the coefficient of $y_1^{n-1} \dots y_{n-k}^{n-1}$ in the polynomial

$$\prod_{i=1}^{n-k} \prod_{j \neq i} (y_i - y_j) \prod_{i=1}^{n-k} \prod_{j=1}^k (y_i - x_j).$$

This means that

$$\frac{d(k, n)}{(n - k)!} = c(k, n),$$

which is the coefficient of $x_1^{n-1} \dots x_k^{n-1}$ in the polynomial

$$P(x_1, \dots, x_k) \prod_{i=1}^k \prod_{j \neq i} (x_i - x_j).$$

The statement of Theorem 1 is obtained as desired.

3. LOCALIZATION IN EQUIVARIANT COHOMOLOGY

In this section, we recall some basic definitions and results in the theory of equivariant cohomology. For more details on this theory, we refer to [1, 2, 3, 5, 6, 9, 10]. Throughout we consider all cohomologies with coefficients in the complex field \mathbb{C} .

Let $T = (\mathbb{C}^*)^n$ be an algebraic torus of dimension n , classified by the principal T -bundle $ET \rightarrow BT$, whose total space ET is contractible. Let X be a compact space endowed with a T -action. Put $X_T = X \times_T ET$, which is itself a bundle over BT with fiber X . Recall that the T -equivariant cohomology of X is defined to be $H_T^*(X) = H^*(X_T)$, where $H^*(X_T)$ is the ordinary cohomology of X_T . Note that $H_T^*(\text{point}) = H^*(BT)$. By pullback via the map $X \rightarrow \text{point}$, we see that $H_T^*(X)$ is an $H^*(BT)$ -module. Thus we may consider $H^*(BT)$ as the coefficient ring for equivariant cohomology.

A T -equivariant vector bundle is a vector bundle E on X together with a lifting of the action on X to an action on E which is linear on fibers. Note that E_T is a vector bundle over X_T .

The T -equivariant Chern classes $c_i^T(E) \in H_T^*(X)$ are defined to be the Chern classes $c_i(E_T)$. If E has rank r , then the top Chern class $c_r^T(E)$ is called the T -equivariant Euler class of E and is denoted $e^T(E) \in H_T^*(X)$. More generally, the T -equivariant characteristic class $c^T(E) \in H_T^*(X)$ is defined to be the characteristic class $c(E_T)$.

Let $\chi(T)$ be the character group of the torus T . For each $\rho \in \chi(T)$, let \mathbb{C}_ρ denote the one-dimensional representation of T determined by ρ . Then $L_\rho = (\mathbb{C}_\rho)_T$ is a line bundle over BT , and the assignment $\rho \mapsto -c_1(L_\rho)$ defines an group isomorphism $f : \chi(T) \simeq H^2(BT)$, which induces a ring isomorphism $\text{Sym}(\chi(T)) \simeq H^*(BT)$. We call $f(\rho)$ the weight of ρ . In particular, we denote by λ_i the weight of ρ_i defined by $\rho_i(x_1, \dots, x_n) = x_i$. We thus obtain an isomorphism

$$H_T^*(\text{point}) = H^*(BT) \simeq \mathbb{C}[\lambda_1, \dots, \lambda_n].$$

Let $\mathcal{R}_T \simeq \mathbb{C}(\lambda_1, \dots, \lambda_n)$ be the field of fractions of $\mathbb{C}[\lambda_1, \dots, \lambda_n]$. An important result in equivariant cohomology is the localization theorem. Historically, localization in equivariant cohomology was studied by Borel [3] and then further investigated by Quillen [11], Atiyah-Bott [1], and Berline-Vergne [2]. Among many versions of the formulation of the localization theorem, we choose the one by Atiyah and Bott [1].

Theorem 5 (Atiyah-Bott [1]). *Let X^T be the fixed point locus of the torus action. Then the inclusion $i : X^T \hookrightarrow X$ induces an isomorphism*

$$i^* : H_T^*(X) \otimes \mathcal{R}_T \simeq H_T^*(X^T) \otimes \mathcal{R}_T.$$

Moreover, Atiyah and Bott [1] also gave an explicit formula for the inverse isomorphism. If X is a compact manifold and X^T is finite, then the localization theorem can be rephrased as follows:

Theorem 6 (Atiyah-Bott [1], Berline-Vergne [2]). *Suppose that X is a compact manifold endowed with a torus action and the fixed point locus X^T is finite. For $\alpha \in H_T^*(X)$, we have*

$$(2) \quad \int_X \alpha = \sum_{p \in X^T} \frac{\alpha|_p}{e_p},$$

where e_p is the T -equivariant Euler class of the tangent bundle at the fixed point p , and $\alpha|_p$ is the restriction of α to the point p .

For many applications, the Atiyah-Bott-Berline-Vergne formula can be formulated in more down-to-earth terms. We are mainly interested in the computation of integrals over Grassmannians.

Proof of Theorem 3. Consider the action of $T = (\mathbb{C}^*)^n$ on \mathbb{C}^n given in coordinates by

$$(a_1, \dots, a_n) \cdot (x_1, \dots, x_n) = (a_1 x_1, \dots, a_n x_n).$$

This induces a torus action on the Grassmannian $G(k, n)$ with isolated fixed points p_I corresponding to coordinate k -planes in \mathbb{C}^n . Each fixed point p_I is indexed by a subset $I \subset [n]$ of size k .

For each p_I , the torus action on the fibers $\mathcal{S}|_{p_I}$ and $\mathcal{Q}|_{p_I}$ have the characters ρ_i for $i \in I$ and ρ_j for $j \in I^c$ respectively. Since the tangent bundle is isomorphic to $\mathcal{S}^\vee \otimes \mathcal{Q}$, the characters of the torus action on the tangent bundle at p_I are

$$\{\rho_j - \rho_i \mid i \in I, j \in I^c\}.$$

Thus the T -equivariant Euler class of the tangent bundle at p_I is

$$e_{p_I} = \prod_{i \in I} \prod_{j \in I^c} (\lambda_j - \lambda_i) = (-1)^{k(n-k)} \prod_{i \in I} \prod_{j \in I^c} (\lambda_i - \lambda_j).$$

By the assumption, the T -equivariant characteristic classes at p_I are

$$\Phi^T(\mathcal{S}|_{p_I}) = P(\lambda_I) \text{ and } \Psi^T(\mathcal{Q}|_{p_I}) = Q(\lambda_{I^c}).$$

By the Atiyah-Bott-Berline-Vergne formula, we have

$$\begin{aligned}
\int_{G(k,n)} \Phi(\mathcal{S}) &= \sum_{p_I} \frac{\Phi^T(\mathcal{S}|_{p_I})}{e_{p_I}} \\
&= (-1)^{k(n-k)} \sum_{I \subset [n], |I|=k} \frac{P(\lambda_I)}{\prod_{i \in I} \prod_{j \in I^c} (\lambda_i - \lambda_j)}, \\
\int_{G(k,n)} \Psi(\mathcal{Q}) &= \sum_{p_I} \frac{\Psi^T(\mathcal{Q}|_{p_I})}{e_{p_I}} \\
&= \sum_{I \subset [n], |I|=k} \frac{Q(\lambda_{I^c})}{\prod_{i \in I} \prod_{j \in I^c} (\lambda_j - \lambda_i)} \\
&= \sum_{I \subset [n], |I|=n-k} \frac{Q(\lambda_I)}{\prod_{i \in I} \prod_{j \in I^c} (\lambda_i - \lambda_j)},
\end{aligned}$$

Combining with Theorem 1, Theorem 3 is proved as desired. \square

Proof of Theorem 4. By the assumption, the T -equivariant characteristic class at p_I is

$$\Delta^T(\mathcal{S}|_{p_I}, \mathcal{Q}|_{p_I}) = P(\lambda_I, \lambda_{I^c}).$$

By the Atiyah-Bott-Berline-Vergne formula, we have

$$\begin{aligned}
\int_{G(k,n)} \Delta(\mathcal{S}, \mathcal{Q}) &= \sum_{p_I} \frac{\Delta^T(\mathcal{S}|_{p_I}, \mathcal{Q}|_{p_I})}{e_{p_I}} \\
&= (-1)^{k(n-k)} \sum_{I \subset [n], |I|=k} \frac{P(\lambda_I, \lambda_{I^c})}{\prod_{i \in I} \prod_{j \in I^c} (\lambda_i - \lambda_j)}.
\end{aligned}$$

Combining with Theorem 2, Theorem 4 is proved as desired. \square

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